of land interpreted in connexion with observations for latitude) involves the unknown error of the chronometer, and makes the ship 1′ West or East of the true place for every four seconds of time that the chronometer's indication is in advance of or behind correct Greenwich time. Although I believe that every man who uses a chronometer at sea knows this perfectly well, I shall not omit to state it in the practical directions which I propose to publish, as the Astronomer Royal, Professor Stokes ('Proceedings,' April 27, 1871), and Mr. Gordon (writing in the 'Mercantile and Shipping Gazette') are of opinion that an explicit warning of the kind might be desirable in connexion with any publication tending to bring Sumner's method into more general use than it has been hitherto.

X. "On Linear Differential Equations."—No. V. By W. H. L. Russell, F.R.S. Received June 15, 1871.

Let us now endeavour to ascertain under what circumstance a linear differential equation admits a solution of the form $P \log_\epsilon Q$, where P and Q are rational functions of (x).

If $(a_0+a_1 x+ \ldots .)\frac{d^n y}{dx^n}+(\beta_0+\beta_1 x+ \ldots .)\frac{d^{n-1}y}{dx^{n-1}}+ \ldots . =0$,

we have, substituting $y=P\log_\epsilon Q$,

$$\left\{(a_0+a_1x+ \ldots .)\frac{d^nP}{dx^n}+(\beta_0+\beta_1x+\ldots .)\frac{d^{n-1}P}{dx^{n-1}}+\ldots\right\}\log_\epsilon Q+R=0,$$

where R is a rational function of (x). Hence

$$(a_0+a_1x+ \ldots .)\frac{d^nP}{dx^n}+(\beta_0+\beta_1x+ \ldots .)\frac{d^{n-1}P}{dx^{n-1}}+ \ldots . =0,$$

or P must be a rational function satisfying the given equation. Having ascertained its value, we have a differential equation of the form

$$L_0\frac{d^n \log_\epsilon Q}{dx^n}+L_1\frac{d^{n-1}\log_\epsilon Q}{dx^{n-1}}+ \ldots . L_n \log_\epsilon Q=0.$$

Divide this equation by L_n and differentiate, and we have an equation of the form

$$M_0\frac{d^{n+1}\log_\epsilon Q}{dx^{n+1}}+M_1\frac{d^n \log_\epsilon Q}{dx^n}+ \ldots . M_n\frac{d\log_\epsilon Q}{dx}=0\,;$$

from which we find Q in all possible cases, since $\frac{d\log_\epsilon Q}{dx}$ is a rational function of (x).

It is impossible that a linear differential equation can in general have a solution of the form $y=f(\log_\epsilon x)$; for in that case we should have

$$(a_0+a_1x+a_2x^2+ \ldots .)\frac{d^nf(\log_\epsilon x)}{dx^n}+(\beta_0+\beta_1x+\beta_2x+ \ldots .)\frac{d^{n-1}f(\log_\epsilon x)}{dx^{n-1}}$$
$$+ \ldots . =0.$$

Let $z=\log_\epsilon x$, and the equation becomes of the form

$$(a_0+a_1e^z+a_2e^{2z}+ \ldots .)\frac{d^nf(z)}{dz^n}+(b_0+b_1e^z+b_2e^{2z}+ \ldots .)\frac{d^{n-1}f(z)}{dz^{n-1}}$$
$$+ \ldots . =0\,;$$

and putting for z successively $z+2\pi i$, $z+4\pi i$, ..., the equation becomes

$$(a_0+a_1e^z+a_2e^{2z}+\ldots.)\frac{d^nf(z+2\pi i)}{dz^n}+(b_0+b_1e^z+b_2e^{2z}+\ldots.$$
$$\frac{d^{n-1}f(z+4\pi i)}{dz^{n-1}}+\ldots.=0,$$
$$(a_0+a_1e^z+a_2e^{2z}+\ldots.)\frac{d^nf(z+4\pi i)}{dz^n}+(b_0+b_1e^z+b_2e^{2z}+\ldots.)$$
$$\frac{d^{n-1}f(z+4\pi i)}{dz^{n-1}}=0,$$
$$\&c. \qquad =\&c.,$$

where these equations can be indefinitely continued.

Let us now see what are the conditions that a linear differential equation can admit of a solution $y=\mathrm{P}+\sqrt[n]{\mathrm{Q}}$, where P and Q are rational functions of (x). It is evident that P and Q must satisfy the differential equation separately, so that we may confine ourselves to the case of $y=\sqrt[n]{\mathrm{Q}}$.

We observe that the factors of Q must also be factors of the coefficient of the highest differential; *i. e.* if

$a_0+a_1x+a_2x^2+\ldots.=(x-a)^m(x-b)^r\ldots.\sqrt[m]{\mathrm{Q}}=(x-a)^\mu(x-b)^\nu\ldots.$

Let $x=a+z$ in the differential equation, and expand y in ascending powers of (z). We have then an equation to determine μ, and ν . . . . may be found in the same way. Let

$$(x+2)(x^2-1)\frac{d^2y}{dx^2}+(x^3+2x^2+2x-2)\frac{dy}{dx}-xy=0.$$

Let $z=x+2$, and the equation becomes

$$z(z^2-4z+3)\frac{d^2y}{dz^2}+(z^3-4z^2+6z-6)\frac{dy}{dz}-(z-2)y=0.$$

Let $y=\mathrm{A}z^n+\mathrm{B}z^{n+1}+\ldots.$, which gives $3n(n-1)-6n=0$, whence $n=0$, or 3.

Let $x=z+1$:

$$z(z+2)(z+3)\frac{d^2y}{dz^2}+(z^3+5z^2+9z+3)\frac{dy}{dz}-(z+1)y=0.$$

Here, putting y as before, we have $6n(n-1)+3n=0$, whence $n=0$, or $\frac{1}{2}$.

Lastly, let $x=z-1$:

$$z(z+1)(z-2)\frac{d^2y}{dz^2}+(z^3-z^2+z-3)\frac{dy}{dz}-(z-1)y=0.$$

Here $2n(n-1)+3n=0$, or $n=0$ or $-\frac{1}{2}$. Hence the possible forms are

$$\sqrt{x-1},\qquad \frac{1}{\sqrt{x+1}},\qquad \text{and}\qquad \frac{\sqrt{x-1}}{\sqrt{x+1}},^*$$

the last of which succeeds.

* And also $(x+2)^3\sqrt{x-1}$, $\dfrac{(x+2)^3}{\sqrt{x+1}}$, $\dfrac{(x+2)^3\sqrt{x-1}}{\sqrt{x+1}}$.—W. H. L. R.—June 30.

I have chosen a particular case, but it is manifest that the equation

$$(\alpha+\beta x+\gamma^{x}+\zeta x^{3})\frac{d^{2}y}{dx^{2}}+(\alpha'+\beta' x+\gamma' x^{2}+\zeta' x^{3})\frac{dy}{dx}+$$
$$(\alpha''+\beta'' x+\gamma'' x^{2}+\zeta'' x^{3})y=0$$

could be treated in the same manner. There will be eight possible forms of solution of the class we have here considered, but in practice the number of trials will be much reduced if we do not consider the incommensurable roots of (n).

XI. "On the Undercurrent Theory of the Ocean, as propounded by recent explorers." By Captain SPRATT, C.B., R.N., F.R.S. Received June 15, 1871.

The universal undercurrent theory so fascinatingly advocated by Maury and others, and more recently by the late Dr. Forchhammer, has now been so remarkably supported and maintained in the enlarged views pronounced by Dr. Carpenter in his recent papers and lectures before the Royal and other societies, and is of so interesting and important a nature in connexion with the study of the laws regulating the natural history and geological results of the past, as well as of the phenomena in progress in the ocean and seas in communication with it, that the assumed facts and data upon which they are based deserve, and indeed require, in the interest of sound science and philosophy, to be carefully considered and analyzed before they can be accepted as a grand law such as is implied in the views or theory.

Dr. Carpenter has put forward certain axioms as "propositions" or fundamental principles, as necessary results from the influence of rivers and rain, temperature, evaporation, and density upon the surface and deeps of all seas that are in communication. I feel it necessary to give the more important of these, as being the basis of his theory*.

"No. III. That wherever there is a *want of equilibrium* arising from *difference of density* between two columns of water in communication with each other, there will be a tendency towards the restoration of equilibrium by a flow from the *lowest stratum* of the denser column towards that of the lighter, in virtue of the excess of pressure to which the former is subjected.

"No. IV. That so long as the like difference of density is maintained, so long will this flow continue; and thus any agency which permanently disturbs the equilibrium in the same sense, either by increasing the density of one column, or by diminishing that of the other, will keep up a permanent flow from the lower stratum of the denser towards that of the less dense. This constant *tendency to restoration of equilibrium* will keep the actual difference of density within definite limits.

"No. V. That if there be at the same time *a difference of level* and an

* See *anteà*, p. 211.